RESEARCH STATEMENT

YEQIN LIU

I am an algebraic geometer who is interested in vector bundles, their moduli spaces, and derived categories of projective varieties.

In algebraic geometry, vector bundles are one of the most well-studied subjects and still are central to modern research. Celebrated achievements include Brill-Noether theory [Laz86], Green's theorem [GL88], and classification of Fano threefolds [Muk92]. Moreover, moduli spaces of vector bundles [Mar77, Mar78] have extremenly rich geometry and give main examples of compact hyperkähler varieties [KLS06, O'G99, PR14]. My research programs leverage techniques from classical algebraic geometry and modern derived category tools to study rigid vector bundles, higher rank Brill-Noether theory, and effective algebraic cycles.

0.1. **Rigid vector bundles.** A vector bundle is *rigid* if it cannot be deformed, and such bundles, especially *exceptional bundles*, are fundamentally important in the studying derived categories, stable vector bundles, and their moduli spaces [BO95, LP97]. However, even in dimensions 2 and 3, classifications of exceptional bundles remain incomplete. For $\mathbb{P}^3$, where attempts at the classification have persisted for over 30 years [Nog91, Rud95], I have recently give the first non-existence result for exceptional bundles.

Theorem 0.1 ([Liu23a])**.** *On $\mathbb{P}^3$, there is no exceptional bundle with:*

1. *(Theorem 1.2) Rank $2d^2+1$ and degree d when $|d| \geq 4$.*
2. *(Theorem 1.4) Rank 27 and degree 11.*

The above result relies on the development of various novel techniques and approaches, resulting in a number of independent directions and applications. For example, *spherical* vector bundles are a fundamental class of rigid bundles on smooth $K3$ surfaces. In my work [Liu23b], I discover that on nodal $K3$ surfaces they exhibit new behaviors (Theorem 1.6). The crucial technical tool for both works [Liu23b, Liu23a] is a way to compute vector bundle cohomology. Although this is a basic invariant of vector bundles and has wide applications [CNY23], it is difficult to compute in general. Using Bridgeland stability [BM14a], I study the cohomology of spherical bundles on $K3$ surfaces and derive the following results.

Theorem 0.2 ([Liu22b])**.** *Let X be a $K3$ surface and E be a stable spherical vector bundle.*

1. *(Theorem 1.7) There is an algorithm to compute the cohomology of E.*
2. *(Theorem 1.8) If $\mathrm{Pic}(X) \cong \mathbb{Z}$, there exists an explicit numerical condition on the Chern class of E that is equivalent to the vanishing of $\mathrm{H}^1(X, E)$.*

In Section 1, I will carefully elaborate the details, new techniques, and many further directions of this program.

0.2. **Brill-Noether theory.** The cohomology jumping loci in a family of vector bundles are called Brill-Noether loci. Classical Brill-Noether theory was built nicely to understand curves better, and motivated the advance of modern algebraic geometry [GH80, FL81, Gie82, EH87]. However in higher ranks and dimensions, many essential questions are still open. I study foundational properties of higher rank Brill-Noether loci on $\mathbb{P}^2$, with a view towards developing the theory to greater generality, see Section 2.

Theorem 0.3 ([GLL22]. See Theorem 2.1)**.** *On $\mathbb{P}^2$, we give a sufficient condition and a necessary condition for the Brill-Noether loci to be non-empty. In degree 1, all Brill-Noether loci are irreducible of expected dimension. In higher degrees, the Brill-Noether loci can have arbitrarily many components of expected or larger dimensions.*

0.3. **Effective cycles.** In the cohomology groups of a variety X, the classes of all dimension k subvarieties in X generate the *cone of effective k-cycles*, which reflect much geometry of X. Although in dimension and codimension one they are well studied [CP90, Kaw84, Kle66], in other dimensions they have come into focus recently [Pet09, Voi10, FL17a, FL17b] and few examples are known. I study cones of effective cycles on blow ups of projective spaces along various curves, and raise a conjecture that is generalizable to other contexts, see Section 3.

Theorem 0.4 ([GL23]. See Theorem 3.1)**.** *For the blow up of $\mathbb{P}^n$ along a curve C:*

1. *When C is a rational normal curve, the cones of effective $1, 2, (n-1)$-cycles are computed. An extremal ray in the cone of effective 3-cycles is found.*
2. *For many other curves C, the cones of effective $1, (n-1)$-cycles are computed.*

0.4. **Ongoing works.** Besides the three mainstreams above, I am also working on various topics recently, including semiorthogonal decompositions of derived categories, intersection theory on stacks, and counting rational points on projective varieties, see Section 4.

1. Rigid vector bundles

My works on rigid vector bundles [Liu23a, Liu23b, Liu22b] are elaborated in the following three sections.

1.1. **Exceptional bundles on $\mathbb{P}^3$.** The goal of this program is to classify exceptional bundles on $\mathbb{P}^3$. In my work [Liu23a], I develop a novel technique for this problem, and give the first non-existence of exceptional bundles on $\mathbb{P}^3$ with certain Chern classes.

On a projective variety X, a vector bundle E is called *exceptional*, if $\mathrm{Hom}_X(E, E) = \mathbb{C}$ and $\mathrm{Ext}^i_X(E, E) = 0$ for all $i > 0$. Exceptional bundles are important in the study of stable sheaves and derived categories. For example, they form components of semi-orthogonal decompositions of derived categories [BO95], and are closely related to the sharp Bogomolov inequalities [DLP85]. To explain my specific goal in this program, the following example illustrates the notion of *constructive* exceptional bundles [Nog91].

Example 1.1. The exceptional bundle $\Omega_{\mathbb{P}^3}$ is obtained by taking the kernel of the evaluation map between the exceptional bundles $\mathcal{O}_{\mathbb{P}^3}(-H)$ and $\mathcal{O}_{\mathbb{P}^3}$ in the following Euler sequence:

$$0 \longrightarrow \Omega_{\mathbb{P}^3} \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-H) \otimes \mathrm{Hom}_{\mathbb{P}^3}(\mathcal{O}_{\mathbb{P}^3}(-H), \mathcal{O}_{\mathbb{P}^3}) \xrightarrow{ev} \mathcal{O}_{\mathbb{P}^3} \longrightarrow 0.$$

The above process is called a *mutation*. Exceptional bundles constructed by iterative mutations from line bundles are called *constructive*.

Unlike on surfaces [Rud90], in higher dimensions we know very little about exceptional bundles. Many important problems are open, among which a major one is whether all exceptional bundles are constructive. This is unknown even on $\mathbb{P}^3$.

Conjecture 1.1 ([Rud95, Nog91])**.** On $\mathbb{P}^3$, all exceptional bundles are constructive.

Let E be an exceptional bundle with rank r and degree d, and $X \subset \mathbb{P}^3$ be a general quartic surface. Since $E|_X$ is a stable spherical vector bundle [Muk87, Kul90b, Zub90], we have $r|(2d^2+1)$. Hence the following is an equivalent way to formulate the Constructivity Conjecture 1.1.

Question 1.1. For $d \in \mathbb{Z}$ and $r > 0$ with $r|(2d^2+1)$, does there exist an exceptional bundle with rank r and degree d?

When $r = 1$, they exist as line bundles. When $r = 2d^2+1$, we only know they exist when $|d| \leq 3$. In the following theorem, I give the first non-existence result on Question 1.1, which shows that $r|(2d^2+1)$ is not the only obstruction.

Theorem 1.2 ([Liu23a])**.** *On $\mathbb{P}^3$, there is no exceptional bundle with rank $2d^2+1$ and degree d for $|d| \geq 4$.*

Although the obstruction used to prove Theorem 1.2 is likely to apply to many other cases, there are various further obstructions. The following phenomenon is such an example.

Phenomenon 1.3. *Let $\{X_t \subset \mathbb{P}^3\}_{t \in L}$ be a general pencil of quartic surfaces and C be the base curve. Let $\mathrm{v} \in \mathrm{H}^*_{alg}(X)$ be a fixed spherical Mukai vector for a general member X and $E_t \in M_{X_t,H}(\mathrm{v})$. Then for general $t \neq t'$ in L, we have $E_t|_C \ncong E_{t'}|_C$.*

In [Liu23a], I show Phenomenon 1.3 actually occurs. Using it, I show the following result which is not covered by Theorem 1.2.

Theorem 1.4 ([Liu23a])**.** *On $\mathbb{P}^3$, there is no exceptional bundle with rank 27 and degree 11.*

Further directions. Understanding stable sheaves on threefolds and their derived categories is an immense and challenging project. One approach, which leads to a triumph on $\mathbb{P}^2$, first studies exceptional bundles and uses them to give obstructions for stable sheaves. Motivated by this, my long term goal is to study sheaves on threefolds by first studying their exceptional bundles. Projects around this topic can be separated in the following two veins.

Classification of exceptional bundles on $\mathbb{P}^3$. This is done if the Constructivity Conjecture 1.1 is proved [Gou23]. My strategy to attack the conjecture, which succeeded in proving Theorem 1.2, consists of the following two steps.

Step 1: Show the following property holds, which is a crucial observation in [Liu23a].

Property 1.5. *Let $X \subset \mathbb{P}^3$ be* any *smooth quartic surface, $\mathrm{v} \in \mathrm{H}^*_{alg}(X)$ be a spherical Mukai vector. Then there exists at most one simple rigid sheaf E on X, such that $E|_C \cong E'|_C$ for any $C = X \cap X'$ and $E' \in M_{X',H}(\mathrm{v})$, where X' is a smooth quartic surface of Picard rank 1.*

Showing Property 1.5 requires a detailed analysis of certain Harder-Narasimhan filtrations, which can be done using techniques developed in my other work [Liu22b]. There, a notion of *height* (Definition 6.13) that measures the complexity of a Chern class is introduced. The analysis successfully handles height 1 Chern classes (Theorem 1.2), and can be pushed further by an argument using the Grauert-Mulich theorem and Bogomolov inequality [HL10]. In the following project, I will carried out the above method to show Property 1.5 for more cases.

Project 1.1. Prove Property 1.5 for other Chern classes. Use the Grauert-Mulich theorem and the Bogomolov inequality to analyze Harder-Narasimhan filtrations, and find a criterion in terms of their shapes. Generalize Theorem 1.2 to height 2 Chern classes.

The objective of Project 1.1 also directly implies the following conjecture, which provides a positive expectation towards the Constructivity Conjecture 1.1 by [BP93, Pol11].

Conjecture 1.2 ([Nog91])**.** On $\mathbb{P}^3$, there is at most one exceptional bundle up to isomorphism with a fixed Chern class.

Step 2: Property 1.5 determines a sheaf, if it exists, on the total family of a pencil of quartics. The second step is to show that this sheaf does not descend to $\mathbb{P}^3$.

If Phenomenon 1.3 happens, then Step 2 is finished. When the Chern class gets larger, I expect Phenomenon 1.3 to occur more frequently. It is likely that the reason lies in a deformation theoretic argument yet to be discovered. I am excited to understand this deeper reason in the following project, aiming to obtain an equivalent condition of Phenomenon 1.3.

Project 1.2. Develop a machinery using deformation theory to study Phenomenon 1.3. Find a sufficient condition for the Chern class such that Phenomenon 1.3 occurs so that more non-existence of exceptional bundles can be showed.

Generalizing known results to other threefolds. In the proof of Theorem 1.2, we covered $\mathbb{P}^3$ by a pencil of quartic surfaces. The choice of this pencil requires a detailed analysis of the Noether-Lefschetz loci of anticanonical surfaces. In the following project, I will carry out this analysis to other Fano threefolds.

Project 1.3. Generalize Theorem 1.2 to other Fano threefolds, by analyzing the Noether-Lefschetz loci of anticanonical $K3$ surfaces.

Theorem 1.1 in [Pol11] gives a sufficient condition for the Uniqueness Conjecture 1.2 on $\mathbb{P}^3$ by using knowledge about sheaves on $K3$ surfaces [Muk87]. On $\mathbb{P}^4$, this is a completely different story, since we know little about sheaves on Calabi-Yau threefolds. Due to the importance of Calabi-Yau threefolds in mathematics and physics, in the long term I would like to contribute to this area by studying their stable sheaves and derived categories. I will consider the following project starting with Calabi-Yau threefolds where stability conditions are constructed.

Project 1.4 (long term)**.** Study stable sheaves on Calabi-Yau threefolds, starting with quintic threefolds [Li19] and $(2,4)$-complete intersections in $\mathbb{P}^5$ [Liu22a]. Study stability of exceptional bundles on $\mathbb{P}^4$. Are their restrictions to a very general quintic threefold stable?

1.2. **Spherical vector bundles on nodal $K3$ surfaces.** In this project [Liu23b], I study spherical vector bundles on nodal $K3$ surfaces with Picard rank 1. I showed that on a nodal $K3$ surface with Picard rank 1, there exists a unique stable spherical vector bundle with any given positive rank spherical Mukai vector.

A projective surface X is called a *nodal $K3$ surface*, if the only singularity of X is a node, $\mathrm{H}^1(X, \mathcal{O}_X) = 0$, and $\omega_X \cong \mathcal{O}_X$. Nodal $K3$ surfaces are the simplest specialization of a $K3$ surface that acquires singularities. Since nodal $K3$ surfaces are K-trivial, they are expected to share many similar properties ([BM14b, KLS06, O'G99, PR14, Yos01]) as $K3$ surfaces. In particular, I am interested in comparing these similarities for stable sheaves and their moduli spaces.

In [Sim94], moduli spaces of semistable sheaves on singular varieties are constructed. Moduli spaces of sheaves on smooth surfaces are extensively studied, however on singular varieties they are less understood. One basic idea is to reduce the study to the smooth case. The two usual approaches are smoothing the singularity or resolving it. For instance, [Zhe23] showed that the Hilbert schemes of surfaces with ADE singularities are irreducible by smoothing the singularities. In [Liu23b], I consider higher rank cases using the second approach. I study spherical vector bundles on nodal $K3$ surfaces by reducing the question to its resolution, which is a smooth $K3$ surface. Using Bridgeland stability, I obtain the following result.

Theorem 1.6. *Let X be a nodal $K3$ surface with $\mathrm{Pic}(X) \cong \mathbb{Z}H$, $\pi : \widetilde{X} \to X$ be its resolution, and $\mathrm{H}^*_{alg}(X)$ be the algebraic cohomology of X. Let $\mathrm{v} = (r, dH, a) \in \mathrm{H}^*_{alg}(X)$ be a spherical Mukai vector with positive rank.*

- *If $\mathrm{Cl}(X) \neq \mathrm{Pic}(X)$ and $r = 2$, then the moduli space of H-Gieseker semistable sheaves $M_{X,H}(\mathrm{v})$ is empty.*
- *Otherwise, $M_{X,H}(\mathrm{v})$ is a reduced point, and the sheaf in $M_{X,H}(\mathrm{v})$ is locally free.*

This project was motivated by Question 1.1. Suppose a spherical Mukai vector v lifts to an exceptional bundle E on $\mathbb{P}^3$, then by a similar argument in [Zub90], $E|_X$ is stable for a general singular quartic surface. However, it was not clear that on a nodal $K3$ surface for which spherical Mukai vectors there exists a stable vector bundle. Theorem 1.6 shows that this is not an obstruction for Question 1.1.

Further directions. In the following I would like to discuss the following two directions: Studying stable sheaves on singular surfaces by a different approach, and generalizing the techniques in [Liu23b] to study other moduli spaces of sheaves on nodal $K3$ surfaces.

In the first direction, our approach to Theorem 1.6 is by resolving the singularity. Another approach to study the moduli spaces of sheaves is to deform the underlying surface so that the singularity is smoothened [Ish00, Ish92]. A surface that I am particularly interested in is a quartic cone in $\mathbb{P}^3$, since the stable rigid sheaves on it are useful in the study of exceptional bundles on $\mathbb{P}^3$ (specifically, see Conjecture 1.2). Another practical example to consider is the quadric cones, since sheaves and their moduli spaces on Hirzebruch surfaces are understood relatively well (e.g. [CH21, ABM12, OU15]). In the following project I will use the deformation theory approach to study moduli spaces of sheaves on singular surfaces.

Project 1.5. Study moduli spaces of stable sheaves on other singular surfaces. For instance, nodal $K3$ surfaces with higher Picard ranks, quartic cones, and quadric cones in $\mathbb{P}^3$. Especially observe their behaviors when the underlying surface deforms to a smooth one.

In the second direction, when the moduli spaces of sheaves on a nodal $K3$ surface have positive dimensions, they have interesting geometry. For instance, on smooth $K3$ surfaces, every moduli space of stable sheaves with respect to a generic polarization is hyperkähler [KLS06, O'G99, PR14]. Since nodal $K3$ surfaces have trivial canonical bundles, these singular moduli spaces are expected to share some similar properties as in the smooth case and have interesting geometry. For example, I expect the moduli spaces of isotropic sheaves on a nodal $K3$ surface X to be nodal $K3$ surfaces which are Fourier-Mukai partners of X. Naturally, we will generalize the counting problem [HLOY03, HLOY04] of Fourier-Mukai partners to this setting. Combining these, in the following project I will study moduli spaces of sheaves on nodal $K3$ surfaces in greater generalility, starting from the Hilbert schemes of points and the moduli spaces of isotropic sheaves.

Project 1.6. Compare the moduli theory on nodal $K3$ surfaces with the case of smooth $K3$ surfaces. What can be said in Hodge theory? Are the moduli spaces of isotropic sheaves nodal

$K3$ surfaces, and are they Fourier-Mukai partners of the underlying surface? Generalize the counting results in [HLOY03, HLOY04] to this setting.

1.3. **Cohomology of spherical vector bundles on K3 surfaces.** In this project, I compuute the cohomology groups of spherical vector bundles on projective $K3$ surfaces. In [Liu22b], I developed an algorithm to compute this invariant for any stable spherical vector bundle on any $K3$ surface. If the Picard rank is 1, I found an explcit numerical condition that is equivalent to the vanishing of h^1 of a spherical vector bundle.

A vector bundle E on a $K3$ surface is called *spherical*, if it is H-stable with respect to some polarization H and $\mathrm{Ext}^1_X(E,E)=0$. Spherical vector bundles are closely related to Bridgeland stability [Huy12]. The moduli spaces of spherical vector bundles are isolated points, hence spherical vector bundles are the simplest vector bundles on $K3$ surfaces and are good for testing new theories. When a $K3$ surface X is embedded into a Fano threefold P, spherical vector bundles on X are also closely related to exceptional bundles on P (see Section 1.1).

Given a vector bundle E, $\mathrm{H}^0(E)$, the space of global sections, is an important invariant of E. On an irreducible moduli space of sheaves, computing $\mathrm{H}^0(-)$ of a general member is the first step of study the Brill-Noether theory for that moduli space. It is also closely related to strange duality [MO13]. However despite being fundamental, this invariant is in general hard to compute. On a $K3$ surface, determining whether a general member of a moduli space has only one non-vanishing cohomology group (called the *weak Brill-Noether* property) is a difficult problem [CNY23], and when weak Brill-Noether fails, computing the exact value of h^0 for a general member is challenging. In [Liu22b], I studied this question completely for spherical vector bundles. The main results are summarized below.

Theorem 1.7 ([Liu22b], Algorithm 3.14)**.** *Let (X,H) be a polarized $K3$ surface, $v\in H^*_{alg}(X)$ a spherical Mukai vector with positive rank, and $E\in M_{X,H}(\mathrm{v})$. There is an algorithm to compute $\mathrm{h}^0(X,E)$ in terms of v.*

Theorem 1.8 ([Liu22b], Theorem 7.1)**.** *Let (X,H) be a polarized $K3$ surface with $\mathrm{Pic}(X)=\mathbb{Z}H$, $\mathrm{v}=(r,dH,a)\in\mathrm{H}^*_{alg}(X), r,d>0$ be a spherical Mukai vector, and $E\in M_{X,H}(\mathrm{v})$. Let y be the largest value of $\frac{a_1d-ad_1}{r_1d-rd_1}$, where $\mathrm{v}_1=(r_1,d_1H,a_1)$ satisfies*

$$\mathrm{v}_1^2=-2,\ \mathrm{v}\mathrm{v}_1<0,\ \frac{ad_1-a_1d}{rd_1-r_1d}<0,\ 0<d_1\le d.$$

Then $\mathrm{H}^1(X,E)=0$ if and only if $y<1$.

Further directions. The two future directions are on the Brill-Noether theory of $K3$ surfaces, and understanding stable rigid bundles on other varieties.

In Brill-Noether theory, the first step is to compute $\mathrm{H}^1(-)$ of a general member in a moduli space. Combining techniques from [Liu22b] and [CNY23], I will generalize Theorem 1.7 to non-spherical Mukai vectors and study its applications in Brill-Noether theory.

My plan is based on the following observation. In Theorem 1.7, the only input is $\Theta=(v,\Gamma,H)$, where $\Gamma=\mathrm{H}^*_{alg}(X)$ is the lattice, $H\in\mathrm{H}^*_{alg}(X)$ is a polarization, and $\mathrm{v}\in\mathrm{H}^*_{alg}(X)$ is a spherical Mukai vector. In other words, for any polarized $K3$ surfaces $(X,H),(Y,H)$ with $\mathrm{H}^*_{alg}(X)\cong\mathrm{H}^*_{alg}(Y)$ preserving H, and a spherical Mukai vector $\mathrm{v}\in\mathrm{H}^*_{alg}(X)$ with positive rank, we have $\mathrm{h}^0(X,E_X)=\mathrm{h}^0(Y,E_Y)$, where $E_X\in M_{X,H}(\mathrm{v})$ and $E_Y\in M_{Y,H}(\mathrm{v})$. For non-spherical Mukai vectors, I expect that the same happens for general members in the moduli spaces. Since the moduli spaces have positive dimensions, the Brill-Noether loci will have interesting geometry. In the following project, I will study the behavior of Brill-Noether loci when the underlying $K3$ surface is varying.

Project 1.7. Study the general cohomology for any stable Mukai vector. Study the Brill-Noether loci of a given lattice-theoretic data Θ, especially observe the behavior of such loci in families. Consider its relation to Hodge theory.

Another application of [Liu22b] is to study stable rigid bundles on other varieties, among which an important class is constructed by mutations from line bundles. In [Kul90a], constructibility of spherical objects on $K3$ surfaces is studied. In [Liu22b], the question is answered for stable spherical vector bundles using wall-crossing techniques on the Bridgeland stability manifold. This also proves the existence of stable spherical vector bundles on $K3$ surfaces differently from the classical specialization approach [Yos01]. On $K3$ surfaces the spherical objects that appeared during the construction are not necessarily sheaves. Motivated by this phenomenon, in the following project I will use a similar method to construct stable rigid bundles on other surfaces, by developing a criterion to detect walls similar to [BM14a].

Project 1.8 (long term)**.** Study stable rigid objects on other surfaces using Bridgeland stability, especially on general type surfaces. Are there exceptional sheaves that are non-constructive by sheaves but constructive by Bridgeland stable objects? Do this on surfaces of high degree in $\mathbb{P}^3$, with a view towards studying exceptional bundles on $\mathbb{P}^3$.

2. Higher rank Brill-Noether theory

In this project [GLL22], I study the goemetry of Brill-Noether loci in the moduli spaces of semistable sheaves on $\mathbb{P}^2$. Specifically, fundamental properties of the Brill-Noether loci including (non)-emptiness, (ir)reducibility, and their dimensions are considered. This is joint work with B. Gould and W. Lee.

Let X be a projective variety and $\mathcal{F}$ a family parametrized by S. By the upper semi-continuity of cohomology, for any integer $k \geq 0$, the locus

$$BN^k(S) := \{s \in S : \mathrm{h}^0(X, \mathcal{F}_s) \geq k\}$$

is a closed subvariety of S. Such loci are called the *Brill-Noether* loci, which are the central objects of study in Brill-Noether theory.

Classical Brill-Noether theory, dating back to the 1870s, studies projective embeddings of curves by studying the cohomology jumping loci in the space of line bundles. Since maps to projective spaces correspond to line bundles, we may take the variety X above to be a smooth curve and S to be its Picard variety. The subject was developed in the 1970s and 80s by Kleiman, Kempf, Griffiths, and Harris [GH80, FL81, Gie82, EH87], and motivated a considerable amount of development of modern algebraic geometry.

There are natural ways to generalize classical Brill-Noether theory: one may take the base variety to be higher dimensional, and take the families of sheaves to have higher rank. Compared to the classical case, these generalizations encounter significant difficulties. One main difference in higher rank is that, unlike the Picard variety, there is no longer a base variety S that parametrize *all* sheaves with a given Chern class. It turns out that a reasonable and important family to consider is the moduli space of (Gieseker) semi-stable sheaves. However, even on surfaces, the moduli spaces of sheaves are much less known than those on curves. On $\mathbb{P}^2$, the moduli spaces of stable sheaves are understood relatively well, and the Brill-Noether loci in them are interesting to consider. The main results of my study is summarized below.

Theorem 2.1 ([GLL22])**.** *Let v be a Chern class on $\mathbb{P}^2$ with positive rank and degree. Let $M(\mathrm{v})$ be the moduli space of semistable sheaves with Chern class* v *and $BN^k(\mathrm{v}) \subset M(\mathrm{v})$ denote the locus of sheaves E with $\mathrm{h}^0(E) \geq k$. Then:*

1. *We have $BN^k(\mathrm{v}) = \emptyset$ for $k > \max\{\mathrm{rk}(\mathrm{v}), (c_1(\mathrm{v})^2 + 3c_1(\mathrm{v})H + 2)/2\}$.*
2. *We have $BN^{\mathrm{rk}(\mathrm{v})} \neq \emptyset$. When $\mathrm{rk} \geq (c_1(\mathrm{v})^2 + 3c_1(\mathrm{v})H + 2)/2$, it contains a component of expected dimension.*
3. *When $c_1(\mathrm{v}) = H$, BN^k is irreducible of expected dimension.*
4. *When $\mu_H(\mathrm{v}) > 1/2$ is not an integer and $\Delta(\mathrm{v}) \gg 0$, $BN^{\mathrm{rk}(\mathrm{v})}(\mathrm{v})$ contains components of dimensions equal and bigger than expected.*

2.1. **Further directions.** My long term goal in this direction is to understand higher rank Brill-Noether theory in higher dimensions. To approach this, I start by the following two practical projects, using derived categories and the coherent systems techniques respectively.

In [GLL22] we discovered an interesting phenomenon in higher rank Brill-Noether theory: the Brill-Noether loci of the minimal degree objects are well-behaved. In Theorem 2.1, the number of components of Brill-Noether loci is unbounded in general, and they can have dimensions bigger than expected. However, when $c_1(\mathrm{v}) = H$, all Brill-Noether loci are irreducible and of expected dimensions. Two closely related works are [ABS14, Fey20], where Mukai's program has been studied and several positive results are deduced. There, a crucial observation is that the Brill-Noether loci are well-behaved for minimal c_1. Since many arguments in our work [GLL22] can be generalized from $\mathbb{P}^2$ to other surfaces and from sheaves to Bridgeland stable objects, in the following project I will study the Brill-Noether theory of the minimal degree objects in greater generality, combining our techniques and Bridgeland stability.

Project 2.1. Study Brill-Noether theory on other surfaces, especially when the degree of the Chern class is minimal. Study Brill-Noether theory for Bridgeland stable objects, observe their behavior under birational modifications of the corresponding wall crossings.

Another perspective to study Brill-Noether theory is to view a general point $[E] \in BN^k(\mathrm{v})$ as a map $\mathcal{O}^k \to E$. Such an object is called a *coherent system*. In [New22], coherent systems are studied on curves, and their applications to higher rank Brill-Noether theory was proved to be very effective. Many techniques are highly generalizable to higher dimensions, where some existing examples are worked out in [LP93]. To approach the long term goal of studying Brill-Noether theory in higher dimensions, I will start by developing moduli theory of coherent systems on surfaces in the following project, to get finer properties of Brill-Noether loci and study their birational geometry by the wall-crossing technique.

Project 2.2 (long term)**.** Develop a theory of coherent systems in higher dimensions, starting on surfaces. Find the appropriate notion of stability and study their moduli spaces. Use the wall-crossing technique to study the birational geoemtry of Brill-Noether loci.

3. Effective cycles

In this project [GL23], I study cones of higher codimension for $\mathrm{BL}_C\mathbb{P}^n$ where $C \subset \mathbb{P}^n$ is a rational normal curve. This is joint work with B. Gould.

The geometry of a subvariety $Y \subset X$ involves the interactions of Y with other subvarieties in X. To some extenet, this can be detected by the subvarieties in the blow up $\mathrm{BL}_Y(X)$. For example, the blow up of a plane at 3 collinear points contains the proper transform of the line containing those points, but this cycle does not exist if the 3 points are in general position. Hence the geometry of $Y \subset X$ is closely related to the cones of effective cycles $\mathrm{Eff}_k(\mathrm{BL}_Y(X)) \subset \mathrm{NS}_k(\mathrm{BL}_Y(X))$. From another perspective, given any variety Z, the cones of effective curves and divisors are widely studied in classical geometry (e.g. [CP90, Kaw84, Kle66, Nag59, Mat02]). Recently, the cones of higher codimension cycles is drawing more attention (e.g. [Pet09, Voi10, DJV13, FL17a, FL17b]), but they are in general difficult to compute, and few examples are known. We obtained the following theorem, and raised a conjecture (Conjecture 3.15 in [GL23]).

Theorem 3.1 ([GL23])**.** *Let $C_d \subset \mathbb{P}^d$ be a rational normal curve, $X_d = \mathrm{BL}_C\mathbb{P}^d$ and $E_d \subset X_d$ be the exceptional divisor. Then*

(1) *The cone $\mathrm{Eff}^1(X_{2n})$ is generated by E and the secant variety of $(n+1)$-secant n-planes. The cone $\mathrm{Eff}^1(X_{2n+1})$ is generated by E and the cone of secant variety of n-planes over any point on C_d.*
(2) *The cone $\mathrm{Eff}_1(X_d)$ is generated by any secant line of C_d and $\mathrm{Eff}_1(E_d)$*
(3) *The cone $\mathrm{Eff}_2(X_d)$ is generated by any 3-secant plane, the cone of C_d over any point on C_d, and $\mathrm{Eff}_2(E_d)$.*
(4) *The cone $\mathrm{Eff}_3(X_d)$ has an extremal ray generated by the secant variety of lines.*

3.1. **Further directions.** In a longer term, my goal is motivated by Conjecture 3.15 in [GL23], which suggests that the geometry of a blow up and the secant variety of the blow up center are related. In Theorem 3.1, we showed that this is true in dimensions 1, 2, 3, and codimension 1. This expectation is also suggested by the work in [Mar17]. In the following project I will study this relation in greater generality.

Project 3.1 (long term)**.** Study the relation between the geometry of a blow up and the secant variety of the blow up center, starting with curves with a well understood embedding (e.g. rational normal curves, canonical curves, complete intersections...), and low dimension secant bundles of surfaces (e.g. secant bundle of lines of the Veronese embeddings of $\mathbb{P}^2$, secant bundle of lines of surfaces in $\mathbb{P}^3$...). Use the technique in [Mar17] to $X_d = \mathrm{BL}_{C_d}\mathbb{P}^n$ to solve the conjecture completely.

To achieve the long term goal above, one approach is to generalize our techniques in [GL23] to other settings. There, the main difficulty is to study the cones of effective cycles secant bundles in greater generality, which shall be my short term goal. Another reason that I am interested in secant bundles is that, in [Ful11] the effective cones of all projective bundles over $\mathbb{P}^1$ are computed. However over higher dimensional bases, they are yet unknown. Secant bundles are good first examples to consider as they arise naturally from geometry. In the following project I will study the cones of effective cycles on secant varieties of various embedded varieties.

Project 3.2. Study cones of effective cycles of secant bundles, starting with the examples in Project 3.1.

4. Ongoing works

In this section I would like to briefly mention some works that I am currently thinking about or planning to do in the near future.

4.1. Phantom category.

Under the mentorship of Daniel Krashen and Alex Perry, Kimoi Kemboi, Tianle Liu, Eoin Mackall, Svetlana Makarova, Antonios-Alexandros Robotis, Sridhar Venkatesh, and I studied phantom categories at MRC 2023 [1].

An admissible subcategory $\mathcal{A}$ of a triangulated category is called a *phantom* if $K_0(\mathcal{A}) = 0$. It was believed that these phantoms cannot arise naturally from geometry [Kuz09] until several examples were discovered recently [GO13, BGvBKS15, Kra23]. Using the notion of normal Hochschild cohomology and height developed by Alexander Kuznestov [Kuz15], people are getting a better understanding of phantoms. Based on the discussion at the MRC 2023, we will study the existence and deformation theory of phantoms on several concrete surfaces, $\mathbb{P}^3$, and certain blow ups of $\mathbb{P}^3$.

4.2. Chen-Ruan Chow ring.

Under the mentorship of Dan Abramovich and Rachel Webb, Shiyue Li, Ming Hao Quek, Shabham Sinha, Hongwei Xu, and I studied the Chen-Ruan Chow ring at AGNES 2023 [2].

The *Chen-Ruan Chow ring*, developed in [CY01b, CY01a], is an intersection theory on Deligne-Mumford stacks that records the information of automorphisms of points. As a generalization of the Chow ring, the Chen-Ruan Chow ring has been studied recently for several examples (e.g. [Jia07, Pag12, Pag13, Per07]). Using techniques discussed at AGNES 2023, we obtained the blow up formula of Chen-Ruan Chow rings under certain conditions (preprint will appear soon), and plan to study the first examples where the stacks are not global quotient of abelian groups.

4.3. The Batyrev-Manin conjecture.

The Batyrev-Manin conjecture [FMT89] predicts the growth of rational points on a variety with respect to a height. Among its various forms and refinements, a simple version can be formulated as follows.

Conjecture 4.1 ((Batyrev-Manin))**.** Let X be a smooth Fano variety over k and $\mathrm{ht}_{-K}(-)$ be the height induced by the anticanonical divisor K_X. We define the rational point counting function for every positive integer B to be

$$N(X, -K_X, B) := \#\{x \in X(k) : \mathrm{ht}_{-K}(x) \leq B\}.$$

Then

$$\lim_{B\to\infty} \frac{\log N(X, -K_x, B)}{\log B} = 1.$$

The conjecture was solved for toric varieties [BT98], and was studied for several examples (e.g. [McK11, LWZ19, ESZB23, Bro07]). However, it is open in general. I plan to study Conjecture 4.1 for several concrete varieties that are rational but not toric. Progress has been made. For instance, inspired by [KT22] I showed that Conjecture 4.1 is true for $(\mathbb{P}^2)^{[4]}$, the Hilbert scheme of 4 points on $\mathbb{P}^2$.

[1] Mathematics Research Communities - 2023.

[2] AGNES Summer School on Intersection Theory on Moduli Spaces 2023.

5. References

[ABM12] M. Aprodu, V. Brînzănescu, and M. Marchitan. Rank-two vector bundles on Hirzebruch surfaces. *Cent. Eur. J. Math.*, 10(4):1321–1330, 2012.

[ABS14] E. Arbarello, A. Bruno, and E. Sernesi. Mukai's program for curves on a K3 surface. *Algebr. Geom.*, 1(5):532–557, 2014.

[BGvBKS15] C. Böhning, H. Graf von Bothmer, L. Katzarkov, and P. Sosna. Determinantal Barlow surfaces and phantom categories. *J. Eur. Math. Soc. (JEMS)*, 17(7):1569–1592, 2015.

[BM14a] A. Bayer and E. Macrì. Mmp for moduli spaces of sheaves on $K3$s via wall-crossing: nef and movable cones, lagrangian fibrations. *Inventiones Mathematicae*, 198(3):505–590, 2014.

[BM14b] A. Bayer and E. Macrì. Projectivity and birational geometry of Bridgeland moduli spaces. *J. Amer. Math. Soc.*, 27(3):707–752, 2014.

[BO95] A. Bondal and D. Orlov. Semiorthogonal decomposition for algebraic varieties. *arXiv:alg-geom/9506012*, 1995.

[BP93] A. I. Bondal and A. E. Polishchuk. Homological properties of associative algebras: the method of helices. *Izv. Ross. Akad. Nauk Ser. Mat.*, 57(2):3–50, 1993.

[Bro07] T. D. Browning. An overview of Manin's conjecture for del Pezzo surfaces. In *Analytic number theory*, volume 7 of *Clay Math. Proc.*, pages 39–55. Amer. Math. Soc., Providence, RI, 2007.

[BT98] V. V. Batyrev and Y. Tschinkel. Manin's conjecture for toric varieties. *J. Algebraic Geom.*, 7(1):15–53, 1998.

[CH21] I. Coskun and J. Huizenga. Existence of semistable sheaves on Hirzebruch surfaces. *Adv. Math.*, 381:Paper No. 107636, 96, 2021.

[CNY23] I. Coskun, H. Nuer, and K. Yoshioka. The cohomology of the general stable sheaf on a K3 surface. *Adv. Math.*, 426:Paper No. 109102, 2023.

[CP90] F. Campana and T. Peternell. Algebraicity of the ample cone of projective varieties. *J. Reine Angew. Math.*, 407:160–166, 1990.

[CY01a] W. Chen and R. Yongbin. A new cohomology theory for orbifold. *arXiv:math/0004129v3*, 2001.

[CY01b] W. Chen and R. Yongbin. Orbifold gromov-witten theory. *arXiv:math/0103156v1*, 2001.

[DJV13] O. Debarre, Z. Jiang, and C. Voisin. Pseudo-effective classes and pushforwards. *Pure Appl. Math. Q.*, 9(4):643–664, 2013.

[DLP85] J.-M. Drezet and J. Le Potier. Fibrés stables et fibrés exceptionnels sur $\mathbf{P}_2$. *Ann. Sci. École Norm. Sup. (4)*, 18(2):193–243, 1985.

[EH87] D. Eisenbud and J. Harris. Irreducibility and monodromy of some families of linear series. *Ann. Sci. École Norm. Sup. (4)*, 20(1):65–87, 1987.

[ESZB23] J. S. Ellenberg, M. Satriano, and D. Zureick-Brown. Heights on stacks and a generalized Batyrev-Manin-Malle conjecture. *Forum Math. Sigma*, 11:Paper No. e14, 54, 2023.

[Fey20] S. Feyzbakhsh. Erratum to Mukai's program (reconstructing a K3 surface from a curve) via wall-crossing (J. reine angew. Math. 765 (2020), 101–137). *J. Reine Angew. Math.*, 768:183, 2020.

[FL81] W. Fulton and R. Lazarsfeld. On the connectedness of degeneracy loci and special divisors. *Acta Math.*, 146(3-4):271–283, 1981.

[FL17a] M. Fulger and B. Lehmann. Kernels of numerical pushforwards. *Adv. Geom.*, 17(3):373–378, 2017.

[FL17b] M. Fulger and B. Lehmann. Positive cones of dual cycle classes. *Algebr. Geom.*, 4(1):1–28, 2017.

[FMT89] J. Franke, Y. I. Manin, and Y. Tschinkel. Rational points of bounded height on Fano varieties. *Invent. Math.*, 95(2):421–435, 1989.

[Ful11] M. Fulger. The cones of effective cycles on projective bundles over curves. *Math. Z.*, 269(1-2):449–459, 2011.

[GH80] P. Griffiths and J. Harris. On the variety of special linear systems on a general algebraic curve. *Duke Math. J.*, 47(1):233–272, 1980.

[Gie82] D. Gieseker. Stable curves and special divisors: Petri's conjecture. *Invent. Math.*, 66(2):251–275, 1982.

[GL88] M. Green and R. Lazarsfeld. Some results on the syzygies of finite sets and algebraic curves. *Compositio Math.*, 67(3):301–314, 1988.

[GL23] B. Gould and Y. Liu. Cones of effective cycles on blow ups of projective spaces along rational curves. *arXiv:2308.13964v1*, 2023.

[GLL22] B. Gould, Y. Liu, and W. Lee. Higher rank brill-noether theory on $\mathbb{P}^2$. *International Mathematics Research Notices*, 2022. rnac333.

[GO13] S. Gorchinskiy and D. Orlov. Geometric phantom categories. *Publ. Math. Inst. Hautes Études Sci.*, 117:329–349, 2013.

[Gou23] B. Gould. Constructive exceptional bundles on $\mathbb{P}^3$. *arXiv:2208.04984*, 2023.

[HL10] Daniel Huybrechts and Manfred Lehn. *The geometry of moduli spaces of sheaves*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 2010.

[HLOY03] S. Hosono, B. H. Lian, K. Oguiso, and S. Yau. Fourier-Mukai partners of a $K3$ surface of Picard number one. In *Vector bundles and representation theory (Columbia, MO, 2002)*, volume 322 of *Contemp. Math.*, pages 43–55. Amer. Math. Soc., Providence, RI, 2003.

[HLOY04] S. Hosono, B. H. Lian, K. Oguiso, and S. Yau. Fourier-Mukai number of a K3 surface. In *Algebraic structures and moduli spaces*, volume 38 of *CRM Proc. Lecture Notes*, pages 177–192. Amer. Math. Soc., Providence, RI, 2004.

[Huy12] D. Huybrechts. Stability conditions via spherical objects. *Math. Z.*, 271(3-4):1253–1270, 2012.

[Ish92] Akira Ishii. On the moduli of reflexive sheaves on a surface with rational double points. *Math. Ann.*, 294(1):125–150, 1992.

[Ish00] A. Ishii. Versal deformation of reflexive modules over rational double points. *Math. Ann.*, 317(2):239–262, 2000.

[Jia07] Y. Jiang. The Chen-Ruan cohomology of weighted projective spaces. *Canad. J. Math.*, 59(5):981–1007, 2007.

[Kaw84] Y. Kawamata. The cone of curves of algebraic varieties. *Ann. of Math. (2)*, 119(3):603–633, 1984.

[Kle66] Steven L. Kleiman. Toward a numerical theory of ampleness. *Ann. of Math. (2)*, 84:293–344, 1966.

[KLS06] D. Kaledin, M. Lehn, and Ch. Sorger. Singular symplectic moduli spaces. *Invent. Math.*, 164(3):591–614, 2006.

[Kra23] J. Krah. A phantom on a rational surface. *arXiv:2304.01269*, 2023.

[KT22] J. L. Kass and F. Thorne. What is the height of two points in the plane? *arXiv:2209.13030v1*, 2022.

[Kul90a] S. A. Kuleshov. An existence theorem for exceptional bundles on k3 surfaces. *Mathematics of the USSR-Izvestiya*, 34(2):373, apr 1990.

[Kul90b] S. A. Kuleshov. Exceptional bundles on $K3$ surfaces. In *Helices and vector bundles*, volume 148 of *London Math. Soc. Lecture Note Ser.*, pages 105–114. Cambridge Univ. Press, Cambridge, 1990.

[Kuz09] A. Kuznetsov. Hochschild homology and semiorthogonal decompositions. *arXiv:0904.4330v1*, 2009.

[Kuz15] A. Kuznetsov. Height of exceptional collections and Hochschild cohomology of quasiphantom categories. *J. Reine Angew. Math.*, 708:213–243, 2015.

[Laz86] R. Lazarsfeld. Brill-Noether-Petri without degenerations. *J. Differential Geom.*, 23(3):299–307, 1986.

[Li19] C. Li. On stability conditions for the quintic threefold. *Invent. Math.*, 218(1):301–340, 2019.

[Liu22a] Shengxuan Liu. Stability condition on Calabi-Yau threefold of complete intersection of quadratic and quartic hypersurfaces. *Forum Math. Sigma*, 10:Paper No. e106, 29, 2022.

[Liu22b] Y. Liu. The cohomology of spherical vector bundles on $k3$ surfaces. *arXiv:2210.11030*, 2022.

[Liu23a] Y. Liu. Nonexistence of exceptional bundles on $\mathbb{P}^3$ with maximal possible ranks. *arXiv:2302.11743v2*, 2023.

[Liu23b] Y. Liu. Spherical vector bundles on nodal k3 surfaces. *arXiv:2307.00091v1*, 2023.

[LP93] J. Le Potier. Systèmes cohérents et structures de niveau. *Astérisque*, (214):143, 1993.

[LP97] J. Le Potier. *Lectures on vector bundles*, volume 54 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1997. Translated by A. Maciocia.

[LWZ19] J. Liu, J. Wu, and Y. Zhao. Manin's conjecture for a class of singular cubic hypersurfaces. *Int. Math. Res. Not. IMRN*, (7):2008–2043, 2019.

[Mar77] M. Maruyama. Moduli of stable sheaves. I. *J. Math. Kyoto Univ.*, 17(1):91–126, 1977.

[Mar78] M. Maruyama. Moduli of stable sheaves. II. *J. Math. Kyoto Univ.*, 18(3):557–614, 1978.

[Mar17] C. Martinez. Duality, Bridgeland wall-crossing and flips of secant varieties. *Internat. J. Math.*, 28(2):1750011, 40, 2017.

[Mat02] K. Matsuki. *Introduction to the Mori program.* Universitext. Springer-Verlag, New York, 2002.

[McK11] D. McKinnon. Vojta's conjecture implies the Batyrev-Manin conjecture for $K3$ surfaces. *Bull. Lond. Math. Soc.*, 43(6):1111–1118, 2011.

[MO13] A. Marian and D. Oprea. Generic strange duality for $K3$ surfaces. *Duke Math. J.*, 162(8):1463–1501, 2013. With an appendix by Kota Yoshioka.

[Muk87] S. Mukai. On the moduli space of bundles on $K3$ surfaces. I. In *Vector bundles on algebraic varieties (Bombay, 1984)*, volume 11 of *Tata Inst. Fund. Res. Stud. Math.*, pages 341–413. Tata Inst. Fund. Res., Bombay, 1987.

[Muk92] S. Mukai. Fano 3-folds. In *Complex projective geometry (Trieste, 1989/Bergen, 1989)*, volume 179 of *London Math. Soc. Lecture Note Ser.*, pages 255–263. Cambridge Univ. Press, Cambridge, 1992.

[Nag59] M. Nagata. On the 14-th problem of Hilbert. *Amer. J. Math.*, 81:766–772, 1959.

[New22] Peter E. Newstead. Higher rank Brill-Noether theory and coherent systems open questions. *Proyecciones*, 41(2):449–480, 2022.

[Nog91] D. Yu. Nogin. Notes on exceptional vector bundles and helices. In *Algebraic geometry (Chicago, IL, 1989)*, volume 1479 of *Lecture Notes in Math.*, pages 181–195. Springer, Berlin, 1991.

[O'G99] Kieran G. O'Grady. Desingularized moduli spaces of sheaves on a $K3$. *J. Reine Angew. Math.*, 512:49–117, 1999.

[OU15] S. Okawa and H. Uehara. Exceptional sheaves on the Hirzebruch surface $\mathbb{F}_2$. *Int. Math. Res. Not. IMRN*, (23):12781–12803, 2015.

[Pag12] N. Pagani. The Chen-Ruan cohomology of moduli of curves of genus 2 with marked points. *Adv. Math.*, 229(3):1643–1687, 2012.

[Pag13] N. Pagani. Chen-Ruan cohomology of $M_{1,n}$ and $\overline{M}_{1,n}$. *Ann. Inst. Fourier (Grenoble)*, 63(4):1469–1509, 2013.

[Per07] F. Perroni. Chen-Ruan cohomology of ADE singularities. *Internat. J. Math.*, 18(9):1009–1059, 2007.

[Pet09] T. Peternell. Submanifolds with ample normal bundles and a conjecture of Hartshorne. In *Interactions of classical and numerical algebraic geometry*, volume 496 of *Contemp. Math.*, pages 317–330. Amer. Math. Soc., Providence, RI, 2009.

[Pol11] A. Polishchuk. Simple helices on Fano threefolds. *Canad. Math. Bull.*, 54(3):520–526, 2011.

[PR14] A. Perego and A. Rapagnetta. Factoriality properties of moduli spaces of sheaves on abelian and K3 surfaces. *Int. Math. Res. Not. IMRN*, (3):643–680, 2014.

[Rud90] *Helices and vector bundles*, volume 148 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 1990. Seminaire Rudakov, Translated from the Russian by A. D. King, P. Kobak and A. Maciocia.

[Rud95] Alexei N. Rudakov. Rigid and exceptional vector bundles and sheaves on a Fano variety. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, pages 697–705. Birkhäuser, Basel, 1995.

[Sim94] Carlos T. Simpson. Moduli of representations of the fundamental group of a smooth projective variety. I. *Inst. Hautes Études Sci. Publ. Math.*, (79):47–129, 1994.

[Voi10] C. Voisin. Coniveau 2 complete intersections and effective cones. *Geom. Funct. Anal.*, 19(5):1494–1513, 2010.

[Yos01] K. Yoshioka. Moduli spaces of stable sheaves on abelian surfaces. *Math. Ann.*, 321(4):817–884, 2001.

[Zhe23] X. Zheng. Irreducibility of the Hilbert schemes of points on surfaces with Kleinian singularities. *Comm. Algebra*, 51(1):98–115, 2023.

[Zub90] D. Yu. Zube. The stability of exceptional bundles on three-dimensional projective space. In *Helices and vector bundles*, volume 148 of *London Math. Soc. Lecture Note Ser.*, pages 115–117. Cambridge Univ. Press, Cambridge, 1990.